LONGWAVE APPROXIMATION IN THE THEORY OF UNIDIRECTIONAL COMPOSITES

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Equilibrium equations for unidirectional composites [2] were derived in [1] on the basis of logical arguments, while the equations of motion of these materials were derived in [3, 4]. These equations are simpler than the exact equations of the theory of elasticity, thus allowing detailed studies to be made of a number of problems concerning the motion and fracture of composites [3-10]. It is shown here that the equations in [1, 3, 4] are a longwave approximation of the equations of the theory of elasticity. It is best to use them if the width of the binder H between reinforcing fibers is small enough compared to the characteristic scale of change in the solution along the reinforcement direction y. This condition is satisfied if the ratio of the stiffness of the binder and reinforcement is sufficiently small. It should be noted that the formulation of the problem of the reinforcement of an elastic medium by fibers which are described by a unidimensional wave equation requires us to assume that the elastic moduli of the fibers are much greater than the moduli of the reinforcement. In fact, when the moduli are the same, we obtain a uniform elastic medium and the unidimensional equations become meaningless. Examples of such materials are plastics reinforced with carbon or boron fibers, rubber-cord conveyor belts, coatings of ice on reservoirs, and reinforcements with steel cables. In the present study, we do not assume that the solution changes slowly in the transition from fiber to fiber. In the latter case, it is possible to regard the composite as a uniform continuum with averaged properties. The theory being proposed here occupies a position which is intermediate between the theory of elasticity of a reinforced medium and the theory of elasticity of a uniform anisotropic medium obtained by "spreading of the structure."

In the formulation of the theory of elasticity below, we solve the problem of the deformation of a unidirectional reinforced medium subjected to body forces. We then obtain the longwave approximation by passing to the limit $qH \to 0$ (where q is the parameter of the Fourier transform for y).

1. Let an elastic plane by reinforced by identical parallel equidistant fibers, and let the body forces $X_j(y, t)$, $Y_j(y, t)$ be distributed along the fibers (where $-\infty < j < \infty$ is the number of the fiber and t is time). We assume that motion begins from a state of rest at $t = -\infty$. In particular, we assume that we can make use of functions of operational calculate, which are equal to zero at $t < 0$. We further suppose that all of the functions we encounter are quadratically summable over j and y and that at $t \to \pm\infty$ their modulus is no greater than the modulus of the function when it is quadratically summable over y, j, and t is multiplied by $\exp(t/\tau)$ (where τ represents positive constants which are the same for all of the functions). Taking such an approach allows us to use the direct and inverse Laplace–Fourier transforms.

We write the Laplace–Fourier transform of the equations which describe flexural and longitudinal vibrations of the j-th fiber in the form

$$\begin{aligned} &q^4 u_j^{LF} + \frac{12}{h^2 c^2} p^2 u_j^{LF} = \frac{12}{h^2}\frac{X_j^{LF}}{E} + \frac{12}{h^3}\frac{[\sigma_{xx}]_j^{LF}}{E}, \\ &q^2 v_j^{LF} + \frac{p^2}{c^2} v_j^{LF} = \frac{Y_j^{LF}}{E} + \frac{[\sigma_{xy}]_j^{LF}}{Eh}. \end{aligned} \tag{1.1}$$

Here, the superscript LF denotes the Laplace–Fourier transforms; p and q are parameters of the transforms; u_j and v_j are the displacements of the j-th fiber along and across the reinforcement direction; c is the velocity of a longitudinal wave in the fibers; E and h are the Young's modulus and the width of a fiber; $[\sigma_{xx}]_j$ and $[\sigma_{xy}]_j$ are the jumps in the shear and normal stresses in the binder with the crossing of the j-th fiber. The stress jumps must be found by solving the system

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$$\frac{\partial^2 u}{\partial x^2} + \frac{1-\nu}{2}\frac{\partial^2 u}{\partial y^2} + \frac{1+\nu}{2}\frac{\partial^2 v}{\partial x \partial y} = \frac{1}{c_1^2}\frac{\partial^2 u}{\partial t^2},$$
$$\frac{\partial^2 v}{\partial y^2} + \frac{1-\nu}{2}\frac{\partial^2 v}{\partial x^2} + \frac{1+\nu}{2}\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{c_1^2}\frac{\partial^2 v}{\partial t^2}, \quad (1.2)$$

which describes the motion of an elastic strip in the case when the displacements of the fibers at its boundaries are assigned.

We will consider a plane stress state, with u and v being the displacements, x the coordinate in the binder perpendicular to the fiber, ν the Poisson's ratio, $(1 - \nu)/2 = c_2^2/c_1^2$, $(1 + \nu)/2 = (c_1^2 - c_2^2)/c_2^2$, c_1, c_2 the velocities of the longitudinal and shear waves in the binder.

Laplace–Fourier transformation of (1.2) leads to the system

$$\frac{d^2 u^{LF}}{dx^2} - \frac{1-\nu}{2} q^2 u^{LF} - \frac{1+\nu}{2} iq \frac{dv^{LF}}{dx} = \frac{1}{c_1^2} p^2 u^{LF},$$
$$\frac{1-\nu}{2}\frac{d^2 v^{LF}}{dx^2} - q^2 v^{LF} - \frac{1+\nu}{2} iq \frac{du^{LF}}{dx} = \frac{1}{c_1^2} p^2 v^{LF}, \quad (1.3)$$

the solution of which has the form

$$\begin{pmatrix} u^{LF} \\ v^{LF} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -i\alpha/q & i\alpha/q & -iq/\beta & iq/\beta \end{pmatrix} \begin{pmatrix} A_1 \exp(\alpha x) \\ A_2 \exp(-\alpha x) \\ A_3 \exp(\beta x) \\ A_4 \exp(-\beta x) \end{pmatrix}, \quad (1.4)$$

where A_j are constants; $\alpha = \sqrt{(p/c_2)^2 + q^2}$; $\beta = \sqrt{(p/c_1)^2 + q^2}$; i is the imaginary unit.

Let us calculate the shear stress on the left edge of the strip (x = 0, j is the number of the fiber):

$$\sigma_{xy}^{LF} = \mu \left(-iqu_j^{LF} + \frac{dv_j^{LF}}{dx}\right) = \mu \left(-2iqu_j^{LF} - \frac{ip^2}{qc_2^2}(A_1 + A_2)\right).$$

The displacements are continuous in the crossing of the j-th fiber, so that the jump in the shear stresses

$$[\sigma_{xy}]_j^{LF} = -\mu \frac{ip^2}{qc_2^2}[A_1 + A_2]_j.$$

The jump of the sum $A_1 + A_2$ with the transition from one layer of binder to another is due to the fact that adjacent layers have different displacements on non-adjacent edges. Similarly, the jump in the normal stresses

$$[\sigma_{xx}]_j^{LF} = \frac{2\mu}{1-\nu}\frac{p^2}{\beta c_1^2}[A_3 - A_4]_j.$$

To find the constants A_j, we use a system of equations in which the right sides are the displacements u_j^{LF}, v_j^{LF}, u_{j+1}^{LF}, v_{j+1}^{LF} on the left and right edges of the strip of binder:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ -\frac{i\alpha}{q} & \frac{i\alpha}{q} & -\frac{i\beta}{q} & \frac{i\beta}{q} \\ e^{\alpha H} & e^{-\alpha H} & e^{\beta H} & e^{-\beta H} \\ -\frac{i\alpha}{q}e^{\alpha H} & \frac{i\alpha}{q}e^{-\alpha H} & -\frac{i\beta}{q}e^{\beta H} & \frac{i\beta}{q}e^{-\beta H} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} = \begin{pmatrix} u_j^{LF} \\ v_j^{LF} \\ u_{j+1}^{LF} \\ v_{j+1}^{LF} \end{pmatrix}. \quad (1.5)$$

The determinant of this system $\Delta = -4/(q^2\beta)\, D(\alpha, \beta)$, where

$$D(\alpha, \beta) = [(\alpha^2\beta^2 + q^4)\,\mathrm{sh}\,\alpha H\,\mathrm{sh}\,\beta H + 2\alpha\beta q^2(1 - \mathrm{ch}\,\alpha H\,\mathrm{ch}\,\beta H)].$$

The stress jumps are linear forms of u_k^{LF}, v_k^{LF} ($k = j-1, j, j+1$). As an example, we will calculate the coefficient with v_j^{LF} in the expression for $\Delta[A_1 + A_2]_j$. Proceedings on the basis of the Kramer rule for the solution of a linear system, we find that it is equal to

$$\begin{vmatrix} 1 & 1 & 1 \\ e^{-\alpha H} & e^{\beta H} & e^{-\beta H} \\ \frac{i\alpha}{q} e^{-\alpha H} & -\frac{iq}{\beta} e^{\beta H} & \frac{iq}{\beta} e^{-\beta H} \end{vmatrix} + \begin{vmatrix} -1 & 1 & 1 \\ -e^{-\alpha H} & e^{\beta H} & e^{-\beta H} \\ \frac{i\alpha}{q} e^{-\alpha H} & -\frac{iq}{\beta} e^{\beta H} & \frac{iq}{\beta} e^{-\beta H} \end{vmatrix} - \cdots$$

The three dots represents the two analogous terms with minus signs. Here, H is replaced by $-H$. Adding the determinants in pairs, we have

$$\begin{vmatrix} 0 & 1 & 1 \\ -2\,\mathrm{sh}\,\alpha H & e^{\beta H} & e^{-\beta H} \\ 2\frac{i\alpha}{q}\,\mathrm{ch}\,\alpha H & -\frac{iq}{\beta} e^{\beta H} & \frac{iq}{\beta} e^{-\beta H} \end{vmatrix} - \begin{vmatrix} 0 & 1 & 1 \\ 2\,\mathrm{sh}\,\alpha H & e^{-\beta H} & e^{\beta H} \\ 2\frac{i\alpha}{q}\,\mathrm{ch}\,\alpha H & -\frac{iq}{\beta} e^{-\beta H} & \frac{iq}{\beta} e^{\beta H} \end{vmatrix}.$$

The second and third columns change places in the second determinant, after which we change the signs in the last row. The two determinants are combined and expanded, giving us

$$\begin{vmatrix} 0 & 1 & 1 \\ -4\,\mathrm{sh}\,\alpha H & e^{\beta H} & e^{-\beta H} \\ 4\frac{i\alpha}{q}\,\mathrm{ch}\,\alpha H & -\frac{iq}{\beta} e^{\beta H} & \frac{iq}{\beta} e^{-\beta H} \end{vmatrix} = \frac{8i}{\beta q}(\alpha\beta\,\mathrm{ch}\,\alpha H\,\mathrm{sh}\,\beta H - q^2\,\mathrm{sh}\,\alpha H\,\mathrm{ch}\,\beta H).$$

We use the same approach to calculate the remaining coefficients with the fiber displacements, and we write the jump in the shear stresses in the form

$$[\sigma_{xy}]_j^{LF} = \mu\frac{p^2}{c_2^2}P(\alpha,\beta)\,(v_{j-1}^{LF} - 2Q(\alpha,\beta)\,v_j^{LF} + v_{j+1}^{LF} + R(\alpha,\beta)\,(u_{j+1}^{LF} - u_{j-1}^{LF})), \tag{1.6}$$

where

$$P(\alpha,\beta) = \beta\,(\alpha\beta\,\mathrm{sh}\,\beta H - q^2\,\mathrm{sh}\,\alpha H)/D(\alpha,\beta); \tag{1.7}$$

$$Q(\alpha,\beta) = \frac{\alpha\beta\,\mathrm{ch}\,\alpha H\,\mathrm{sh}\,\beta H - q^2\,\mathrm{sh}\,\alpha H\,\mathrm{ch}\,\beta H}{\alpha\beta\,\mathrm{sh}\,\beta H - q^2\,\mathrm{sh}\,\alpha H}; \tag{1.8}$$

$$R(\alpha,\beta) = iq\,\frac{\alpha\,(\mathrm{ch}\,\beta H - \mathrm{ch}\,\alpha H)}{\alpha\beta\,\mathrm{sh}\,\beta H - q^2\,\mathrm{sh}\,\alpha H}. \tag{1.9}$$

We similarly obtain the jump in the normal stresses

$$[\sigma_{xx}]_j^{LF} = \frac{2\mu}{1-\nu}\frac{p^2}{c_1^2}P(\beta,\alpha)\,(u_{j-1}^{LF} - 2Q(\beta,\alpha)\,u_j^{LF} + u_{j+1}^{LF} - R(\beta,\alpha)\,(v_{j+1}^{LF} - v_{j-1}^{LF})). \tag{1.10}$$

As should be expected, the coefficient with u_j^{LF} in the expression for $[\sigma_{xy}]_j^{LF}$ is equal to zero. It is zero because the horizontal displacement of the j-th fiber does not lead to a jump in the shear stresses on the fiber. For similar reasons, the term with v_j^{LF} is omitted for $[\sigma_{xx}]_j^{LF}$.

If we insert the resulting stress jumps into (1.1), we obtain an infinite system of linear equations in the fiber displacements u_j^{LF}, v_j^{LF}. The displacements of the binder can be determined from (1.4), (1.5) by making the right side of (1.5) the displacements of the fibers bounding the strip of binder.

The infinite system we obtain is reduced to two equations by discrete Fourier transformation with respect to the index j. Here we multiply each equation of (1.1) by exp (ijs), where $-\pi < s < \pi$ is the transform parameter. We then sum the equations of each group over j from $-\infty$ to $+\infty$. In discrete Fourier transformation of the stress jumps, it should be considered that a sum of the form $(v_{j+1}^{LF} + v_{j-1}^{LF})$ becomes $2v^{LFF}\cos s$, while a difference of the form $(v_{j+1}^{LF} - v_{j-1}^{LF})$ becomes $-2iv^{LFF}\sin s$. The inversion formula express the coefficient v_j^{LF} of the Fourier series in terms of its sum $v^{LFF}(s)$:

$$v_j^{LF} = \frac{1}{2\pi}\int_{\pi}^{\pi} v^{LFF}(s)\exp(-ijs)\,ds.$$

Having represented the stress jumps as

$$\begin{pmatrix} [\sigma_{xx}]^{LFF} \\ [\sigma_{xy}]^{LFF} \end{pmatrix} = \mu \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} u^{LFF} \\ v^{LFF} \end{pmatrix}, \tag{1.11}$$

where

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \dfrac{4p^2 P(\beta, \alpha)}{(1-\nu)c_1^2} & 0 \\ 0 & \dfrac{2p^2}{c_2^2}P(\alpha, \beta) \end{pmatrix} \begin{pmatrix} \cos s - Q(\beta, \alpha) & iR(\beta, \alpha)\sin s \\ -iR(\alpha, \beta)\sin s & \cos s - Q(\alpha, \beta) \end{pmatrix},$$

and having inserted (1.11) into (1.1), we arrive at the system

$$\begin{pmatrix} q^4 + \dfrac{12}{h^2c^2}p^2 - \dfrac{12}{h^3}\dfrac{\mu}{E}a_{11} & -\dfrac{12}{h^3}\dfrac{\mu}{E}a_{12} \\ -\dfrac{1}{h}\dfrac{\mu}{E}a_{21} & q^2 + \dfrac{p^2}{c^2} - \dfrac{\mu}{Eh}a_{22} \end{pmatrix} \begin{pmatrix} u^{LFF} \\ v^{LFF} \end{pmatrix} = \begin{pmatrix} \dfrac{12}{h^2}\dfrac{X^{LFF}}{E} \\ \dfrac{Y^{LFF}}{E} \end{pmatrix}. \tag{1.12}$$

The determinant of this system is nontrivial. This follows from the uniqueness of the solution, which can deduced as usual from the law of conservation of energy. It follows from the uniqueness of the solution, which can deduced as usual from the law of conservation of energy. It follows from an analysis of the explicit expressions above that the matrix – changing the body forces X^{LFF}, Y^{LFF} into fiber displacements u^{LFF}, v^{LFF} – consists of bounded functions which decrease relatively rapidly as p, q $\to \infty$. It also follows from the analysis that there are inverse transformations that can be used to obtain the originals $u_j(y, t)$, $v_j(y, t)$. The latter, together with the derivatives in the equation of motion, belong to the class described at the beginning of this article. Thus, the above formal calculations are shown to be valid.

We can use the approach described above or we can pass to the limit $p \to 0$ in (1.1), (1.6-1.10) to obtain a system of equations for the fiber displacements u_j^F, v_j^F in the static case. As in the dynamic case, the static stress jumps have the form (1.7), (1.9), but the functions $P(\alpha, \beta)$, $Q(\alpha, \beta)$, $R(\alpha, \beta)$ change to the form:

$$P(\alpha, \beta) = 2\frac{q}{p^2}c_1^2c_2^2\frac{(c_2^2 - c_1^2)qH\,\mathrm{ch}\,qH + (c_2^2 + c_1^2)\,\mathrm{sh}\,qH}{(c_1^2 + c_2^2)^2\,\mathrm{sh}^2\,qH - (c_1^2 - c_2^2)^2q^2H^2}; \tag{1.13}$$

$$Q(\alpha, \beta) = \frac{\mathrm{sh}\,qH\,\mathrm{ch}\,qH\,(c_1^2 + c_2^2) + (c_2^2 - c_1^2)qH}{(c_1^2 + c_2^2)\,\mathrm{sh}\,qH + (c_2^2 - c_1^2)qH\,\mathrm{ch}\,qH}; \tag{1.14}$$

$$R(\alpha, \beta) = \frac{iqH\,\mathrm{sh}\,qH\,(c_2^2 - c_1^2)}{\mathrm{sh}\,qH\,(c_1^2 + c_2^2) + (c_2^2 - c_1^2)qH\,\mathrm{ch}\,qH}. \tag{1.15}$$

The transportation of α and β in expression (1.10) for the jump $[\sigma_{xx}]_j^{LF}$ now means that c_1 and c_2 also change places. With passage to the limit in (1.8-1.10), it becomes necessary to replace α, β by their approximate expansions in powers of p:

$$\alpha \cong q\left(1 + \frac{p^2}{2c_2^2q^2} - \frac{p^4}{8c_2^4q^4}\right), \quad \beta \cong q\left(1 + \frac{p^2}{2c_1^2q^2} - \frac{p^4}{8c_1^4q^4}\right).$$

2. Analyzing the solution of equations (1.12) is a fairly difficult task. The elastic waves in the binder – which interacts with the fibers – generate a complex pattern. In most cases, the details of this pattern have no effect on the qualitative properties (strength or wave-guide properties, for example) of the composite. The situation is similar to the kinetic theory of gases, where it is not only impossible to know the behavior of each particle, but also unnecessary.

We will simplify the problem by making use of a circumstance which is always true of actual composites – the elastic modulus of the reinforcement is much greater than the shear modulus of the binder, i.e. $\mu/E << 1$.

Let us see how the stress jumps behave when $\mu/E \to 0$. An analysis of system (1.12) shows that if $\left(q^4 + \frac{12}{h^2c^2} p^2\right) \left(q^2 + \frac{p^2}{c^2}\right)$ is nontrivial, then stress jump (1.11) approaches zero as $\mu/E \to 0$. If $q^2 + 12(p^2/c^2)$ is equal to zero, then the stress jump $[\sigma_{xy}]^{LFF}$ approaches a nontrivial finite value as $\mu/E \to 0$. Here, the dependence of the jump on q is equal to the product of $\sqrt{\mu/E}$ and the δ function localized near $q^2 = -p^2/c^2$. If $q^4 + (12/h^2c^2)p^2$ is equal to zero, then $[\sigma_{xx}]^{LFF}$ approaches a nontrivial limit.

Thus, for small μ/E, the stress jumps are localized near $q^2 = -p^2/c^2$, $q^4 = -12p^2/h^2c^2$. As a result, for a sufficiently small ratio μ/E we can replace the coefficients a_{1j} in (1.11) by their values with $q^4 = -12p^2/h^2c^2$. In this case, we also replace the coefficients a_{2j} by their values with $q^2 = -p^2/c^2$. This gives the correct limiting values of the stress jumps when $\mu/E \to 0$, $\left(q^4 + \frac{12}{h_2^2c_2^2} p^2\right) \left(q^2 + \frac{p^2}{c_2^2}\right) = 0$. The error caused by such a substitution when the given product is nontrivial is negligible, since a_{ij} enters into system (1.12) in the form of a product with a small factor μ/E. The determinant of the system is nontrivial when $\mu/E = 0$. Thus, after such a substitution, the ratios c_1^2/c^2, c_2^2/c^2 – which are proportional to μ/E – appear in a_{ij} and in P, Q, and R from (1.7-1.9). We will assume that the ratios of the densities of the binder and reinforcement are such that the ratios of the velocities c_1/c and c_2/c approach zero as $\mu/E \to 0$. (As a rule, the densities of the fibers and binder in a composite are similar, and the given velocity ratios actually approach zero.) Thus, in (1.7-1.9) we can ignore the squares of these ratios relative to unity. This is formally equivalent to passing to the limit $q \to 0$.

Consequently, when $\mu/E \to 0$, $c_1/c \to 0$, and $c_2/c \to 0$, the stress jumps are localized near $q = 0$. We will pass to the limit $q \to 0$ in (1.7-1.9). This leads to an approximate theory which is naturally called the longwave approximation. It is understood that the passage to the limit $q \to 0$ applies only to a_{ij}, not to all of the other functions. In particular, it does not apply to the fiber displacements in (1.6), (1.10).

In (1.7-1.9), we multiply the numerator and denominator of the fractions by powers of H such that we obtain the combinations αH, βH, qH and pass to the limit at $qH \to 0$. We will retain the same notation for the limiting values. As a result, instead of (1.7-1.9) we have

$$\begin{aligned} &\alpha H = pH/c_2, \quad \beta H = pH/c_1, && R(\alpha, \beta) = R(\beta, \alpha) = 0, \\ &P(\alpha, \beta) = \frac{c_2}{p \operatorname{sh}(pH/c_2)}, && P(\beta, \alpha) = \frac{c_1}{p \operatorname{sh}(pH/c_1)}, \\ &Q(\alpha, \beta) = \operatorname{ch}(pH/c_2), && Q(\beta, \alpha) = \operatorname{ch}(pH/c_1). \end{aligned} \tag{2.1}$$

Instead of (1.6-1.10), we obtain expressions for the stress jumps

$$\begin{aligned} [\sigma_{xy}^{LF}]_j &= \mu \frac{p}{c_2} \frac{1}{\operatorname{sh}(pH/c_2)} (v_{j+1}^{LF} - 2 \operatorname{ch}(pH/c_2) v_j^{LF} + v_{j-1}^{LF}), \\ [\sigma_{xx}^{LF}]_j &= \frac{2\mu}{1-\nu} \frac{p}{c_1} \frac{1}{\operatorname{sh}(pH/c_1)} (u_{j+1}^{LF} - 2 \operatorname{ch}(pH/c_1) u_j^{LF} + u_{j-1}^{LF}), \end{aligned} \tag{2.2}$$

the first of which was obtained in [3, 4].

It can be seen from (1.12) and (2.1) that, in the longwave approximation, the problem of reinforcement motion breaks down into two independent problems on the horizontal and vertical displacements. In the exact formulation, $R \neq 0$ and these displacements affect one another.

To find the displacement of the binder in the longwave approximation, we need to have expressions for the constants A_i from system (1.5). To avoid having to perform lengthy calculations in order to retain the information lost in passing to the limit $qH \to 0$, we first simplify the system itself. We can then easily calculate the unknowns

$$\begin{aligned} &A_1 = \frac{q}{2i\alpha \operatorname{sh} \alpha H} (e^{-\alpha H} v_j^{LF} - v_{j+1}^{LF}), \quad A_2 = \frac{q}{2i\alpha \operatorname{sh} \alpha H} (e^{\alpha H} v_j^{LF} - v_{j+1}^{LF}), \\ &A_3 = \frac{q}{2 \operatorname{sh} \beta H} (-e_j^{-\beta H} u^{LF} + u_{j+1}^{LF}), \quad A_4 = \frac{q}{2 \operatorname{sh} \beta H} (e^{\beta H} u_j^{LF} - u_{j+1}^{LF}), \end{aligned}$$

where α and β represent their limiting values, pH/c_2, pH/c_1 (2.1). We insert the values found for A_i into (1.4). The horizontal and vertical motions of the binder also turn out to be independent:

$$\mathrm{sh}\,(pH/c_1)\,u^{LF} = -u_j^{LF}\,\mathrm{sh}\,(p\,(x-H)/c_1) + u_{j+1}^{LF}\,\mathrm{sh}\,(px/c_1),$$
$$\mathrm{sh}\,(pH/c_2)\,v^{LF} = -v_j^{LF}\,\mathrm{sh}\,(p\,(x-H)/c_2) + v_{j+1}^{LF}\,\mathrm{sh}\,(px/c_2). \quad (2.3)$$

The transforms u^{LF} and v^{LF} in (2.3) consist of terms of the form f(p) exp (px/c). These terms describe stationary waves moving to the right and left with the velocity c. In fact, if we apply the Laplace transform to the wave F(x − ct) moving to the right, we obtain a transform of the required form

$$\int_{-\infty}^{\infty} F\,(x - ct)\,\exp\,(-pt)\,dt = f\,(p)\,\exp\,(-px/c).$$

Thus, in the longwave approximation, the displacements of the binder are described by unidimensional wave equations

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c_1^2}\frac{\partial^2 u}{\partial t^2},\quad \frac{\partial^2 v}{\partial x^2} = \frac{1}{c_2^2}\frac{\partial^2 v}{\partial t^2},\quad c_2^2 = \frac{\mu}{\rho},\quad c_1^2 = \frac{2}{1-\nu}c_2^2 \quad (2.4)$$

(μ being the shear modulus of the binder and ρ its density).

Conversely, we can find u^{LF} and v^{LF} by subjecting (2.4) to integral transformations and assigning the displacements of the fibers as boundary conditions for (2.4). This was the approach used in [3, 4], and it gives the displacements of the binder (2.3) and the stress jumps (2.2). It is also possible to obtain (2.4) from system (1.3) by setting q = 0 in the latter and returning to the space of the originals.

The above simplification is expedient if the contribution to the Fourier−Laplace integral changing the transform back into the original is concentrated in the region $|qH| << 1$. In this case, the solution does not undergo any abrupt changes along y at lengths on the order of H. This is a standard condition of use of approximate thin-body theories. In particular, reinforcing fibers can be regarded as unidimensional rods when the solution does not undergo any abrupt changes at lengths on the order of the transverse dimension h of the fibers. Since it is usually the case that H ≈ h in composites, the conditions for use of the theory being proposed here to explain binder behavior are nearly the same as for the equations of motion of the reinforcement.

Let us examine two special cases in which the longwave approximation coincides with the exact theory. Let the fibers be rigid rods. This means that the motion of each fiber is composed of rotational and transnational components. We will assume that the rotational component is absent due to the character of the external load. Then the boundary conditions for the binder strip, i.e. for system (1.2), will be independent of y. Thus, the solution will also be independent of y. As a result, if we eliminate the derivatives with respect to y we obtain (2.4) for the displacements of the binder in the given case. For the same reasons, Eqs. (2.4) are also satisfied exactly when the stiffness of the fibers is arbitrary but the load X, Y is independent of y. It is understood that the fibers are fractured by infinite forces in such cases.

Equations (2.4) are likely to adequately describe the behavior of the binder if the fibers are sufficiently rigid compared to the binder and if the external loads do not change too rapidly along the fibers. The absence of derivatives with respect to y in (2.4) does not mean that the displacements u, v of the binder are independent of y. This dependence remains in force in the solution due to the boundary conditions at the points of attachment of the binder and fibers. It is important only that the change of the function u, v along y not be too rapid.

In the static case, by reasoning as above we find that if the parameter μH/Eh characterizing the ratio of the stiffness of the reinforcement and binder is small enough, then the stress jumps will be localized at q = 0 and the longwave approximation (qH → 0 in (1.6), (1.10), (1.13-1.15)) will lead to the following expressions for the stress jumps [1]:

$$[\sigma_{xy}]_j^F = \mu\,\frac{1}{H}\,(v_{j+1}^F - 2v_j^F + v_{j-1}^F),$$
$$[\sigma_{xx}]_j^F = \frac{2\mu}{1-\nu}\,\frac{1}{H}\,(u_{j+1}^F - 2u_j^F + u_{j-1}^F).$$

These formulas can also be obtained from (2.2) by passing to the limit p → 0. The displacements of the binder in the static case are linear functions of the coordinate x. The parameters c_1/c and c_2/c are no longer important.

3. Let us discuss considerations of a mechanical nature which justify the replacement of (1.2) by (2.4). Since the given simplifications make use of the character of elastic interaction between the components, they cannot be obtained from a separate examination of Eqs. (1.2). We will suppose that there is a rectangular element containing fibers and binder. The total forces on the horizontal area (perpendicular to the fibers) is composed of the forces in the fibers and in the binder, since the fibers and binder are "parallel-connected." Due to the bonding of the components and the stiffness of the fibers compared to the fibers, we can assume that the entire normal load is concentrated in the fibers. For the same reasons, we can also assume that the tangential force is localized in the fibers in the form of a shearing force. Supposing in the equations of motion of the binder that the stress on the horizontal areas is equivalent to zero ($\sigma_{xy} = 0$ for motion in the x direction and $\sigma_{yy} = 0$ for motion in the y direction), we obtain the equations of motion of the binder in abbreviated form

$$\frac{\partial \sigma_{xx}}{\partial x} - \rho \frac{\partial^2 u}{\partial t^2} = 0, \quad \frac{\partial \sigma_{xy}}{\partial x} - \rho \frac{\partial^2 v}{\partial t^2} = 0. \tag{3.1}$$

Examining the forces on the vertical areas, we assume that the fibers and binder are "series-connected." Thus, the stresses σ_{xx} and σ_{xy} in the binder on these areas may be quite large, despite the low stiffness of the binder – since its strains are not constrained by the fibers. It is precisely through these stresses that the interaction of the fibers takes place; their omission yields a set of isolated fibers, rather than a composite working as a whole. Similar considerations were discussed in [1, 2] for the static case. However, they are by themselves inadequate to derive (2.4). To obtain (2.4), we need to supplement the above equations of motion with the abbreviated form of Hooke's law

$$\sigma_{xx} = \frac{2\mu}{1-\nu} \frac{\partial u}{\partial x}, \quad \sigma_{xy} = \mu \frac{\partial v}{\partial x}, \tag{3.2}$$

which is obtained by dropping the derivatives with respect to y from the complete form of the law. This approach is valid only if we assume that the displacements along y change slowly, as is characteristic of the longwave approximation. Use of the above considerations regarding the forces leads to another system of equations of binder motion. This system is obtained by taking the stresses from the abbreviated Hooke's law and inserting them into (3.1):

$$\frac{\partial^2 u}{\partial x^2} + \frac{1+\nu}{2} \frac{\partial^2 v}{\partial x \partial y} = \frac{1}{c_1^2} \frac{\partial^2 u}{\partial t^2}, \quad \frac{\partial^2 v}{\partial x^2} + \frac{1+\nu}{2} \frac{\partial^2 u}{\partial x \partial y} = \frac{1}{c_2^2} \frac{\partial^2 v}{\partial t^2}. \tag{3.3}$$

System (3.3) can be useful for describing a stress state that changes rapidly along the reinforcement direction, such as near the tip of a crack. Comparison of (3.3) and (1.3) shows that the derivation of (3.3) entails dropping of the terms with q^2 from (1.3), retention of the terms of first-order smallness with respect to q, and a return to the space of the originals. Thus, with respect to q, (3.3) more closely approximates the exact theory of elasticity than (2.4). This leads to the idea of obtaining increasingly accurate approximations by replacing P, Q, and R from (1.7-1.9) by their Taylor expansions in the neighborhood of the point $q = 0$. A more accurate approximation might also be obtained if we solve system (1.3) by the method of expansion in powers of the small parameter q.

Ignoring the tangential stresses on the horizontal areas and accounting for them on the vertical areas disturbs the pairing of these stresses in the binder, which is based on the law of conservation of momentum. However, this law is satisfied for an element of the composite containing fibers and binder, since the tangential forces are transferred from the binder to the fibers and become a shearing force. Thus, the binder can no longer be considered a continuous two-dimensional elastic medium. Instead, it must be regarded as a set of infinitely thin plates which transmit longitudinal and shear waves in the direction of the x axis. These plates do not interact directly with one another. They interact with each other only through the fibers to which they are attached.

Thus, similar results are obtained through logical reasoning in the space of the originals and the formal procedure of changing to the longwave approximation in the transform space. The formal derivation is not rigorous, so that differentiation of the original with resect to y corresponds to multiplication of the Fourier–Laplace transform by $-iq$. As a result, the order of smallness of the corresponding term in (1.3) at $q \to 0$ turns out to be greater than that of the remaining terms. Discarding of the derivatives with respect to y in space of the original corresponds to discarding of terms with the factors q and q^2 in (1.3).

4. It follows from the arguments in Part 2 that at

$$\mu/E \to 0, \quad c_1/c \to 0, \quad c_2/c \to 0 \tag{4.1}$$

the transforms of the stress jumps approach the transforms (2.2) of the stress jumps in the longwave approximation at each point. The same conclusion can be made in regard to the stresses in the solution of (1.1). However, convergence of the transforms does not necessarily imply convergence of the originals (the Laplace transform $\omega/(p^2 + \omega^2)$ of the function, equal to zero at $t < 0$ and $\sin \omega t$ at $t > 0$, approaches zero as $\omega \to \infty$, but the original does not approach zero). On the other hand, changing over to the longwave approximation in the transform space is equivalent to discarding terms including the small parameters q, q^2 with the lower derivatives in system (1.3). Thus, as $q \to 0$ the solution of (1.3) approaches the solution of the abbreviated system of the longwave approximation. Under conditions (4.1), the solution in the formulation of the theory of elasticity is localized near $q = 0$ (see Part 2), and this ensures convergence to the longwave approximation in the transform space. In the space of the originals, changing from (1.2) to (2.4) entails discarding certain higher derivatives. It is therefore not clear whether or not its solution will approach the solution of the abbreviated system if (4.1) is satisfied. Thus, convergence of the longwave solution to the exact solution in the space of the originals required special study. All of the above regarding the dynamic case also applies to the static formulation if (4.1) is replaced by the condition $\mu H/Eh \to 0$.

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